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ORIGINAL RESEARCH

Upper bound of fractional differential operator related to univalent functions

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Abstract In this article, we defined the generalized fractional differential Tremblay operator in the open unit disk that by usage the definition of the generalized Srivastava–Owa operator. In particular, we established a new operator denoted by $\Theta_z^{\beta,\tau,\gamma}$ based on the normalized generalized fractional differential operator and represented by convolution product. Moreover, we studied the coefficient criteria of univalence, starlikeness and convexity for the last operator mentioned.

Keywords Univalent function · Subclasses of univalent function · Generalized fractional differential operator · Convolution

Mathematics Subject Classification 30C45 · 30C55

Introduction

Let $\mathcal{A}(m)$ denoted the class of functions $\psi(z)$ of the form:

$$\psi(z) = z + \sum_{\kappa=m+1}^{\infty} a_\kappa z^\kappa \tag{1}$$

which are analytic and univalent functions in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

and we set $\mathcal{A}(1) \equiv \mathcal{A}$ when $m = 1$. Let $\mathcal{S}(m)$ denoted the subclass of $\mathcal{A}(m)$ representing of all the univalent functions (or schlict or one-to-one) in $\mathbb{U}$ for $\psi(0) = 0$ and $\psi'(0) = 1$. The functions $\psi(z) \in \mathcal{A}(m)$ are said to be the starlike functions of order $\lambda\,(0 \le \lambda < 1)$ in $\mathbb{U}$, if it satisfies the form

$$\Re\left\{\frac{z\psi'(z)}{\psi(z)}\right\} > \lambda \quad (z \in \mathbb{U}), \tag{2}$$

we note that $\mathcal{S}_\lambda^*(m) \subseteq \mathcal{S}_0^*(m) \equiv \mathcal{S}^*(m) \subseteq \mathcal{S}(m)$. Otherwise, The functions $\psi(z) \in \mathcal{S}(m)$ are said to be convex functions of order$\lambda\,(0 \le \lambda < 1)$, if it satisfies the form

$$\Re\left\{\frac{z\psi''(z)}{\psi'(z)} + 1\right\} > \lambda \quad (z \in \mathbb{U}) \tag{3}$$

which symbolized by $\mathcal{K}_\lambda(m) \subseteq \mathcal{K}_0(m) \equiv \mathcal{K}(m)$ and $\mathcal{K}_\lambda(m) \subseteq \mathcal{S}_\lambda^*(m)$. The classes $\mathcal{S}_\lambda^*(m)$ and $\mathcal{K}_\lambda(m)$ have been discussed by many researchers (see [1, 2]). For $m = 1$, the classes $\mathcal{K}_\lambda(1)$ and $\mathcal{S}_\lambda^*(1)$ of order $\lambda\,(0 \le \lambda < 1)$ were studied before by Robertson [3], and by setting $\lambda = 0$, they are represented as equivalent form:

$$\mathcal{K}_\lambda(1) \equiv \mathcal{K}_\lambda \subseteq \mathcal{K}_0 \equiv \mathcal{K} \quad \text{and} \quad \mathcal{S}_\lambda^*(1) \equiv \mathcal{S}_\lambda^* \subseteq \mathcal{S}_0^* \equiv \mathcal{S}^*.$$

Theorem 1 (Bieberbach's Conjecture [4, 5]) *The functions $\psi(z)$ which is defined in (1), is the univalent function in class $\mathcal{S}(1)$, if $|a_\kappa| \le \kappa$ for all $\kappa \ge 2$ and its convex functions in the class $\mathcal{K}(1)$ if $|a_\kappa| \le 1$.*

Next, the concept of convolution (or Hadamard product) for two analytic and univalent functions $\psi(z)$ given by (1) and $h(z) = z + \sum_{\kappa=m+1}^{\infty} b_\kappa z^\kappa$, $m = \{1, 2, 3, \ldots\}$ defined by

$$\psi * h(z) = z + \sum_{\kappa=m+1}^{\infty} a_\kappa b_\kappa z^\kappa. \tag{4}$$

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Let us here recall some the well known geometric properties for the convolution (or Hadamard product) due to Ruscheweyh (see [6]).

Lemma 1 *From* [6, 7], *we have*

(1) *For the functions* $\psi(z)$ *and* $h(z) \in \mathcal{A}(m)$, *and* c *a constant, then we have*
$$c\big(\psi * h\big)(z) = c\,\psi * h(z) = \psi * c\,h(z).$$

(2) *The derivative convolution of two functions belong to the class* $\mathcal{A}(m)$ *is defined as:*
$$\begin{aligned} z(h*\psi)'(z) &= \; h * z\psi'(z) \\ &= \; z + \sum_{\kappa=m+1}^{\infty} \kappa a_\kappa b_\kappa z^\kappa. \end{aligned}$$

(3) *Let the functions* $\psi(z) \in \mathcal{S}^*(m)$ *and* $h(z) \in \mathcal{K}(m)$, *then* $(\psi * h)(z) \in \mathcal{S}^*(m)$.

(4) *For each functions* $\psi(z)$ *and* $h(z) \in \mathcal{K}(m)$, *then* $(\psi * h)(z) \in \mathcal{K}(m)$.

In [8, 9], Srivastava and Owa defined the fractional integral and differential operators in the complex z-plane $\mathbb{C}$ as the formula:

Definition 1 The fractional integral of order σ is defined, for a function $f(z)$ by:

$$I_z^\sigma f(z) := \frac{1}{\Gamma(\sigma)} \int_0^z f(\zeta)(z-\zeta)^{\sigma-1} \mathrm{d}\zeta, \tag{5}$$

where $0 \le \sigma < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\sigma-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2 The fractional derivative of order σ is defined, for a function $f(z)$, by

$$D_z^\sigma f(z) := \frac{1}{\Gamma(1-\sigma)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z f(\zeta)(z-\zeta)^{-\sigma} \mathrm{d}\zeta, \tag{6}$$

where $0 \le \sigma < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\sigma}$ is removed, same in the Definition (1) above.

Tremblay defined one of the successful fractional operators in [10]. Recently, some geometric properties and applications for Termblay's operator $\mathfrak{T}_z^{\beta,\gamma}$ in complex plane and in particular on the open unit disk $\mathbb{U}$, studied and discussed by [11–13].

Definition 3 For $0 < \beta \le 1, 0 < \tau \le 1$ and $1 > \beta - \tau \ge 0$. The Tremblay operator $\mathfrak{T}_z^{\beta,\tau} f(z)$ of function $f(z) \in \mathcal{A}(1)$, for all $z \in \mathbb{U}$ is defined as:

$$\mathfrak{T}_z^{\beta,\tau} f(z) := \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D^{\beta-\tau} z^{\beta-1} f(z), \quad (z \in \mathbb{U}). \tag{7}$$

Example 1 We find the fractional derivative Termblay operator $\mathfrak{T}_z^{\beta,\tau} f(z)$ in Definition 3, where the function $f(z) = z^\mu$, and $\mu \in \mathbb{R}$.

$$\mathfrak{T}_z^{\beta,\tau}\{z^\mu\} = \frac{\Gamma(\tau)\Gamma(\mu+\beta)}{\Gamma(\beta)\Gamma(\mu+\tau)}\{z^\mu\},$$

if $\mu = 1$, we have

$$\mathfrak{T}_z^{\beta,\tau}\{z\} = \frac{\Gamma(\tau)\Gamma(1+\beta)}{\Gamma(\beta)\Gamma(1+\tau)}\{z\}$$

and, if $\beta, \tau = 1$, then

$$\mathfrak{T}_z^{1,1}\{z^\mu\} = \{z^\mu\}. \tag{8}$$

Ibrahim defined a generalization of the fractional differential and integral Srivastava–Owa operators in the open unit disk $\mathbb{U}$ as follows [14]:

Definition 4 If $0 \le \alpha < 1$, $\eta \ge 0$, then defined the generalized fractional integral Srivastava–Owa operator of order α such as

$$\mathcal{I}_z^{\alpha,\eta} f(z) := \frac{(\eta+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\eta+1} - \zeta^{\eta+1})^{\alpha-1} \zeta^\eta f(\zeta) \mathrm{d}\zeta, \tag{9}$$

where $f(z)$ function is analytic in simply-connoted region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z^{\eta+1} - \zeta^{\eta+1})^{-\alpha}$ is removed by requiring $\log(z^{\eta+1} - \zeta^{\eta+1})$ to be real when $(z^{\eta+1} - \zeta^{\eta+1}) > 0$.

Definition 5 If $0 \le \alpha < 1$, $\eta \ge 0$, then defined the generalized fractional derivative Srivastava–Owa operator of order α such as

$$\mathcal{D}_z^{\alpha,\eta} f(z) := \frac{(\eta+1)^{\alpha}}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z (z^{\eta+1} - \zeta^{\eta+1})^{-\alpha} \zeta^\eta f(\zeta) \mathrm{d}\zeta, \tag{10}$$

where $0 \le \alpha < 1$, $\eta > 0$ and $f(z)$ function is analytic in simply-connoted region of the complex z-plane C containing the origin, and the multiplicity of $(z^{\eta+1} - \zeta^{\eta+1})^{-\alpha}$ is removed , as in Definition 4 above.

Lemma 2 *Let* $f(z) \in \mathcal{A}$; *for all* $z \in \mathbb{U}$, $\rho \in \mathbb{R}$, $0 \le \alpha < 1$ *and* $\eta \ge 0$, then

$$\mathcal{D}_z^{\alpha,\eta}\{z^\rho\} = \frac{(\eta+1)^{\alpha-1}\Gamma\left(\frac{\rho}{\eta+1}+1\right)}{\Gamma\left(\frac{\rho}{\eta+1}+1-\alpha\right)} z^{(1-\alpha)(\eta+1)+\rho-1}$$

and

$$\mathcal{I}_z^{\alpha,\eta}\{z^\rho\} = \frac{(\eta+1)^{-\alpha}\Gamma\left(\frac{\rho+\eta+1}{\eta+1}\right)}{\Gamma\left(\frac{\rho+\eta+1}{\eta+1}\right)} z^{\alpha(\eta+1)+\rho}.$$

Next, we included the Fox–Wright function, which is one of the special functions that generalize hypergeometric functions (see [10]), let denoted this function by ${}_p\Lambda_q$ and defined as:

$$\begin{aligned} {}_p\Lambda_q\left(\begin{matrix}(\rho_1, A_1), \ldots, (\rho_p, A_p); \\ \\ (\lambda_1, B_1), \ldots, (\lambda_q, B_q);\end{matrix} \; z\right) &:= \sum_{\kappa=0}^{\infty} \frac{\Gamma(\rho_1, \kappa A_1)\ldots\Gamma(\rho_p, \kappa A_p)}{\Gamma(\lambda_1, \kappa B_1)\ldots\Gamma(\lambda_q, \kappa B_q)} \frac{z^\kappa}{(1)_\kappa} \\ &= \sum_{\kappa=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\rho_i, \kappa A_i)}{\prod_{j=1}^{q} \Gamma(\lambda_j, \kappa B_j)} \frac{z^\kappa}{(1)_\kappa}. \end{aligned}$$

In particular, $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, then they turn into (see [15, 16])

$$\begin{aligned} {}_p\Lambda_q\left(\begin{matrix}(\rho_1, 1), \ldots, (\rho_p, 1); \\ \\ (\lambda_1, 1), \ldots, (\lambda_q, 1);\end{matrix} \; z\right) &= \left[\frac{\prod_{i=1}^{p} \Gamma(\rho_i)}{\prod_{j=1}^{q} \Gamma(\lambda_j)}\right] {}_pF_q(\rho_1, \ldots\rho_p; \lambda_1, \ldots, \lambda_q), \end{aligned}$$

where ρ_i, λ_j are parameters in complex plan $\mathbb{C}$. $A_i > 0$, $B_j > 0$ for all $j = 1, \ldots, q$ and $i = 1, \ldots, p$, such that $0 \le 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i$ for fitting values $|z| < 1$. For all $z \in \mathbb{C}$ and $\kappa \in \{2, 3, 4, \ldots\}$, the Pochhammer symbol $(z)_\kappa$ defined as:

$$(z)_0 = 1 \quad \text{and} \quad (z)_\kappa = z(z+1)\ldots(z+\kappa-1) \quad (\kappa \in N). \tag{11}$$

where $(z)_\kappa = \frac{\Gamma(z+\kappa)}{\Gamma(z)}$ and the formula $\Gamma(z)$ is the well known gamma function. In fact, this function have many remarkable properties in complex plan, we here review some of them. For $z \in \mathbb{C}$, then

$$\Gamma(z+1) := z\Gamma(z). \tag{12}$$

and

$$z\Gamma(z-1) := \Gamma(z) \quad (z > 0). \tag{13}$$

Moreover, we consider the Bloch space $\mathbb{B}(\mathbb{U})$ of all functions analytic and univalent functions f in $\mathcal{A}$ which is defined as [17]:

$$\|f\|_{\mathbb{B}} = \sup_{z\in\mathbb{U}} \left(1 - |z|^2\right) |f'(z)| < \infty, \quad z(\in \mathbb{U}). \tag{14}$$

In the present paper, the generalized Tremblay operator with univalent function $\mathcal{S}(1)$, which is considered as the generalized fractional derivative operator in Definition 5, was defined. After ward, we utilized the normalized generalized Tremblay operator in a class of analytic functions $\mathcal{A}(m)$, with subclasses $\mathcal{S}(m), \mathcal{S}_\lambda^*(m)$ and $\mathcal{K}_\lambda(m)$ in the open unit disk. Furthermore, we performed some applications to prove the bound coefficient for the last operator.

Results

In this section, we defined the generalized fractional differential of the Tremblay operator in Definition 6 according to definition of the generalized fractional derivative of the Srivastava–Owa operator in complex plane $\mathbb{C}$, for the special case, $m = 1$ in classes $\mathcal{A}$ and $\mathcal{S}$. Examples of power function in complex z-plane and some boundedness properties in Bloch space for the operator mentioned were presented as well.

Definition 6 Let $0 \le \beta \le 1$, $0 \le \tau \le 1$ and $\gamma \ge 0$. The generalized fractional differential Tremblay operator of two parameters, is defined as

$$\mathfrak{T}_z^{\beta,\tau,\gamma} f(z) := \frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)\Gamma(1-\beta-\tau)} \left(z^{1-\tau}\frac{\mathrm{d}}{\mathrm{d}z}\right) \int_0^z \frac{\zeta^{\gamma+\beta-1} f(\zeta)}{(z^{\gamma+1} - \zeta^{\gamma+1})^{\beta-\tau}} \mathrm{d}\zeta, \tag{15}$$

where the function $f(z)$ is analytic and univalent in simple-connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z^{\gamma+1} - \zeta^{\gamma+1})^{-\beta+\tau}$ is removed by requiring $\log(z^{\gamma+1} - \zeta^{\gamma+1})$ to be non-negative when $(z^{\gamma+1} - \zeta^{\gamma+1}) > 0$.

Next, we provided a survey of the interest operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ to satisfy a boundedness property in the open unit disk and gave an example by using Definition 6. Note that proving the boundedness operator on Bloch space requires using expression (1), when $m = 1$.

Example 2 Let $f(z) := z^\kappa$, $z \in \mathbb{U}$ and $\kappa \in N$. If $0 < \beta \le 1$, $0 < \tau \le 1$, $\gamma \ge 0$, and $0 \le \beta - \tau < 1$, then the generalized of Termblay operator with power function satisfy

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z^\kappa\} := \frac{(\gamma+1)^{\beta-\tau}\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)\Gamma(\tau)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)\Gamma(\beta)} z^{(1-\beta+\tau)\gamma+\kappa},$$

note here, if $\kappa = 1$, we obtain

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z\} := \frac{(\gamma+1)^{\beta-\tau}\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma(\tau)}{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma(\beta)} z^{(1-\beta+\tau)\gamma+1},$$

now go back to Example 1, we see that, if $\gamma = 0$, then

$$\mathfrak{T}_z^{\beta,\tau,0}\{z^\kappa\} := \frac{\Gamma(\tau)}{\Gamma(\beta)}\frac{\Gamma(\kappa+\beta)}{\Gamma(\kappa+\tau)}\{z^\kappa\}$$

and if $\beta = 1, \tau = 1, \gamma = 0$, we get

$$\mathfrak{T}_z^{1,1,0}\{z^\kappa\} := \{z^\kappa\}.$$

In next theorem, we considered the form of definition of the power series to prove the operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ is bounded with the univalent function $\mathcal{S}$ on Bloch space $\mathbb{B}(\mathbb{U})$ in the open unit disk.

Theorem 2 *Let the function $f \in \mathcal{S}(1) \equiv \mathcal{S}$ belongs to $\mathbb{U}$. Then the operator $\mathfrak{T}_z^{\beta,\tau,\gamma} : \mathcal{S} \to \mathcal{S}$ is bounded on the Bloch $\mathbb{B}(\mathbb{U})$, if*

$$\| \mathfrak{T}_z^{\beta,\tau,\gamma} f \|_{\mathbb{B}} \leq M \ \| f \|_{\mathbb{B}}$$

where

$$M := \frac{r^{(1+\tau-\beta)\gamma}(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)} {}_2\Lambda_1(r)$$

Proof By supposing *f(z)* in class of $\mathcal{S}$, we employ Lemma 1 and Example 2, we obtain

$$\begin{aligned}
&\|\mathfrak{T}_z^{\beta,\tau,\gamma} f(z)\|_{\mathbb{B}} \\
&= (1-|z|^2)\left|\left(\mathfrak{T}_z^{\beta,\tau,\gamma} f(z)\right)'\right| \\
&= (1-|z|^2)\left|\left(\frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D_z^{\beta-\tau,\gamma} z^{\beta-1} f(z)\right)'\right| \\
&= (1-|z|^2)\left|\left(\frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D_z^{\beta-\tau,\gamma} z^{\beta-1}\left\{\sum_{\kappa=0}^{\infty}(1)_\kappa a_\kappa \frac{z^\kappa}{\kappa!}\right\}\right)'\right| \\
&= (1-|z|^2)\left|\left(\frac{\Gamma(\tau)}{\Gamma(\beta)}\sum_{\kappa=0}^{\infty}\frac{(\gamma+1)^{\beta-\tau}\Gamma(\kappa+1)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1+\tau-\beta\right)}\frac{a_\kappa}{(1)_\kappa} z^{(1+\tau-\beta)\gamma+\kappa}\right)'\right|
\end{aligned}$$

since $|z| < r$, for all $z \in \mathbb{U}$, then

$$\begin{aligned}
&\| \mathfrak{T}_z^{\beta,\tau,\gamma} f(z) \|_{\mathbb{B}} \\
&\leq (1-|r|^2)\left|\left(\frac{r^{(1+\tau-\beta)\gamma}(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)}\sum_{\kappa=0}^{\infty}\frac{\Gamma(\kappa+1)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1+\tau-\beta\right)} a_\kappa \frac{r^\kappa}{(1)_\kappa}\right)'\right| \\
&= (1-|r|^2)\left|\left(\frac{r^{(1+\tau-\beta)\gamma+1}(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)} {}_2\Lambda_1(r) * f(r)\right)'\right| \\
&= (1-|r|^2)\left|\frac{r^{(1+\tau-\beta)\gamma}(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)} {}_2\Lambda_1(r) * f'(r)\right|, \\
&= M\|f\|_{\mathbb{B}}.
\end{aligned}$$

where $M := \frac{r^{(1+\tau-\beta)\gamma}(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)} {}_2\Lambda_1(r)$ and

$${}_2\Lambda_1(r) := {}_2\Lambda_1\left(\begin{matrix}(1,1), \left(1+\frac{\beta}{\gamma+1}-\frac{1}{\gamma+1}, \frac{1}{\gamma+1}\right); \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1}-\frac{1}{\gamma+1}, \frac{1}{\gamma+1}\right);\end{matrix} \quad r\right).$$

Normalized operator

In this section we defined a new operator in Theorem 3, which is normalized for the generalized Tremblay operator $\mathfrak{T}_z^{\beta,\tau,\gamma} f(z)$ with an analytic function in the class $\mathcal{A}(m)$.

Theorem 3 *Let the following conditions to be realized:*

$$0 \leq \beta - \tau < 1, \quad \gamma \geq 0. \tag{16}$$

Then the normalized of generalized Tremblay operator in Definition 6 is denoted by $\Theta_z^{\beta,\tau,\gamma} f(z)$ and defined as:

$$\Theta_z^{\beta,\tau,\gamma} f(z) = z + \sum_{\kappa=m+1}^{\infty} \vartheta_{\beta,\tau,\gamma}(\kappa) a_\kappa z^\kappa \ \ m \in \{1,2,3,\ldots\}. \tag{17}$$

For all $f(z) \in \mathcal{A}(m)$ and $|z| < 1$, where

$$\vartheta_{\beta,\tau,\gamma}(\kappa) := \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}. \tag{18}$$

Proof From Definition 6, and by considering the function

$$h(z) = \frac{z^{(\beta-\tau-1)\gamma}\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma(\beta)}{(\gamma+1)^{\beta-\tau}\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma(\tau)},$$

we have

$$\begin{aligned}
&h(z)\, \mathfrak{T}_z^{\beta,\tau,\gamma} f(z) \\
&= \frac{z^{(\beta-\tau-1)\gamma}\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma(\beta)}{(\gamma+1)^{\beta-\tau}\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma(\tau)} \\
&\quad \times \left(\frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D_z^{\beta-\tau,\gamma}\left\{z^\beta + \sum_{\kappa=m+1}^{\infty} a_\kappa z^{\kappa+\beta-1}\right\}\right)
\end{aligned} \tag{19}$$

then

$$\begin{aligned}
&= \frac{z^{(\beta-\tau-1)\gamma}\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma(\beta)}{(\gamma+1)^{\beta-\tau}\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma(\tau)} \\
&\left\{\frac{(\gamma+1)^{\beta-\tau-1}\Gamma(\tau)\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)} z^{(1-\beta+\tau)\gamma+1}\right. \\
&\quad \left. + \sum_{\kappa=m+1}^{\infty}\frac{(\gamma+1)^{\beta-\tau-1}\Gamma(\tau)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma(\beta)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)} a_\kappa z^{(1-\beta+\tau)\gamma+\kappa}\right\}
\end{aligned}$$

which equals to

$$= z + \sum_{\kappa=m+1}^{\infty} \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)} a_\kappa z^\kappa$$

$$= z + \sum_{\kappa=m+1}^{\infty} \vartheta_{\beta,\tau,\gamma}(\kappa)\, a_\kappa z^\kappa. \tag{20}$$

Thus, the normalized for the generalized Tremblay operator is represented as the power series and preserves the class $\mathcal{A}(m)$ with their subclasses, where $m = 1, 2, \ldots$ in the open unit disk $\mathbb{U}$, as

$$\Theta_z^{\beta,\tau,\gamma} f(z) := h(z)\mathfrak{T}_z^{\beta,\tau,\gamma} f(z).$$

□

Lemma 3 Let the operator $\Theta_z^{\beta,\tau,\gamma} f(z)$ defined in the class $\mathcal{S}(m)$, $m \in N\backslash\{0\}$, for all $z \in \mathbb{U}$. Then

$$r = \left(\lim_{\kappa\to\infty} |a_\kappa|^{1/\kappa} |\vartheta_{\beta,\tau,\gamma}(\kappa)|^{1/\kappa}\right) \le 1.$$

Proof By employing the Cauchy–Hadamard formal, we find the radius of convergence of the series function in $\Theta_z^{\beta,\tau,\gamma} f(z)$. Supposing the function $f(z) \in \mathcal{S}(m)$ then the coefficient $|a_\kappa| \le \kappa$ for $\kappa \in N = \{2, 3, 4, \ldots\}$ through Theorem 1, we see that

$$\lim_{\kappa\to\infty} |a_\kappa|^{1/\kappa} \le \lim_{\kappa\to\infty} |\kappa|^{1/\kappa} \le 1,$$

and

$$\lim_{\kappa\to\infty} |\vartheta_{\beta,\tau,\gamma}(\kappa)|^{1/\kappa}$$

$$= \lim_{\kappa\to\infty} \left(\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\right)^{1/\kappa} \left(\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\right)^{1/\kappa}$$

By using the property of gamma function, we have

$$\frac{\Gamma\left(\frac{\kappa}{\gamma+1}+\frac{\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa}{\gamma+1}+\frac{\beta-1}{\gamma+1}+1-\beta+\tau\right)} \sim \kappa^{\beta-\tau}, \quad \kappa\to\infty$$

$$= \lim_{\kappa\to\infty} \left(\left(\frac{\kappa}{\gamma+1}\right)^{1/\kappa}\right)^{\beta-\tau}$$

$$= 1$$

thus follows $r \le 1$. □

Criteria for Hadamard product

In this section, the operator in (17) is represented as the convolution product of two univalent functions in class of $\mathcal{S}(m) \in \mathbb{U}$, in particular, when $m = 1$.

Theorem 4 *Let $f \in \mathcal{S}(1) \equiv \mathcal{S}$ be an univalent function in $\mathbb{U}$. Then we appear the operator $\Theta_z^{\beta,\tau,\gamma}$ as the convolution of two functions in $\mathcal{S}$,*

$$\Theta_z^{\beta,\tau,\gamma} f(z) = g(z) * f(z)$$

where $g(z) := \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} {}_2\Lambda_1(z)$.

Proof By equality (20), we have

$$\Theta_z^{\beta,\tau,\gamma} f(z) = z + \sum_{\kappa=2}^{\infty} \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)} a_\kappa z^\kappa$$

$$= \sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa+1)\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)} a_\kappa \frac{z^\kappa}{\kappa!}$$

$$= \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}$$

$${}_2\Lambda_1\left(\begin{array}{c} (1,1), \left(1+\frac{\beta}{\gamma+1}-\frac{1}{\gamma+1}, \frac{1}{\gamma+1}\right); \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1}-\frac{1}{\gamma+1}, \frac{1}{\gamma+1}\right); \end{array} z\right) * f(z) \tag{21}$$

hence

$$\Theta_z^{\beta,\tau,\gamma} f(z) := \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} {}_2\Lambda_1(z) * f(z)$$

by letting

$$g(z) = \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} {}_2\Lambda_1(z). \tag{22}$$

Then the proof is completed. □

Based on the results, the following observations were obtained. Let $f(z) \in \mathcal{A}(1) \subseteq \mathbb{U}$.

1. If $\beta, \tau = 1, \gamma = 0$, then $\Theta_z^{1,1,0} f(z) := f(z)$ defined in (1), for $m = 1$.
2. If $\gamma = 0$, then $\Theta_z^{\beta,\tau,0} f(z) := \mathfrak{T}_z^{\beta,\tau} f(z)$ defined in (7), for $m = 1$.
3. It is clear that the operator $\Theta_z^{\beta,\tau,\gamma} f(z)$ is generalized of Carlson–Shaffer operator, when $\gamma = 0$ and $\frac{\tau}{\beta} = 1$ in (21), while the linear operator of Carlson and Shaffer defined as [18]:

   $$\mathcal{L}(a,c)f(z) = \varphi(a,c;z) * f(z), f \in \mathcal{A}$$

   where $\varphi(a,c;z) = \sum_{\kappa=0}^{\infty} \frac{(a)_\kappa}{(c)_\kappa} z^\kappa$, $z \in \mathrm{U}, a \in R$, $c \in \{1, 2, 3, \ldots\}$.

Note here the proof of the following Theorems comes immediately from Eq. (20) and Lemma 1.

Theorem 5 *Let* $0<\beta\leq 1$, $0<\tau\leq 1$ *and the condition* (16). *If the function f(z) given by* (1) *in the class* $\mathcal{S}^*(m)$ *and the function* $g(z)$ *defined by* (22) *in* $\mathcal{K}(m)$. *Then*

$f(z)*g(z)\in\mathcal{S}^*(m)$.

Theorem 6 *Let* $0<\beta\leq 1$, $0<\tau\leq 1$ *and the condition* (16). *If the functions f(z) given by* (1) *and g(z) defined by* (22) *in* $\mathcal{K}(m)$. *Then*

$f(z)*g(z)\in\mathcal{K}(m)$.

Univalency of the operator $\Theta_z^{\beta,\tau,\gamma}$

We discussed the initialization of a univalent criteria and convexity by employing the normalized Tremblay operator in the open unit disk, in particular when $m=1$.

Theorem 7 *Let* $f\in\mathcal{S}(1)$. *If the following conditions satisfied*

(i) *for* $0<\beta\leq 1$, $0<\tau\leq 1$ *such that* $0\leq\beta-\tau<1$.
(ii) $0<\rho_i$, $i=1,\ldots,p$ *and* $0<\lambda_j$, $j=1,\ldots,q$; $p\leq q+1$,

then the operator $\Theta_z^{\beta,\tau,\gamma}f(z)\in\mathcal{S}$ *in open unite disk* $\mathbb{U}$.

$$
{}_2\Lambda_1\left(\begin{matrix}(3,1),\left(1+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \qquad\qquad\qquad\qquad\qquad\qquad 1 \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1}\right);\end{matrix}\right)
+{}_2\Lambda_1\left(\begin{matrix}(2,1),\left(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \qquad\qquad\qquad\qquad\qquad 1 \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\beta+1}\right);\end{matrix}\right)
<2\left(\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}\right).
$$

Proof By supposing the function $f\in\mathcal{S}$ with equality (17), we have

$$\Theta_z^{\beta,\tau,\gamma}f(z)=z+\sum_{\kappa=2}^{\infty}w_\kappa z^\kappa,$$

where $w_\kappa:=\vartheta_{\beta,\tau,\gamma}(\kappa)a_\kappa$ and the function $\vartheta_{\beta,\tau,\gamma}(\kappa)$ defined in (18) satisfied the following condition in class $\mathcal{S}$ as follows:

$$\ell_1;=\sum_{\kappa=2}^{\infty}\kappa\,|w_\kappa|=\sum_{\kappa=2}^{\infty}\kappa\,\vartheta_{\beta,\tau,\gamma}(\kappa)\,|a_\kappa|<1,$$

By using Theorem 1, we give the estimate for the coefficients of an univalent function belong to $\mathcal{S}$ in $\mathbb{U}$ also, by employ this estimate, we can get another estimate for ℓ_1 in $\mathcal{S}$ as follows,

$$
\begin{aligned}
\ell_1&=\sum_{\kappa=2}^{\infty}\kappa\vartheta_{\beta,\tau,\gamma}|a_\kappa|\\
&\leq\sum_{\kappa=2}^{\infty}\kappa^2\vartheta_{\beta,\tau,\gamma}(\kappa)=\sum_{\kappa=2}^{\infty}\frac{(\kappa)^2}{\kappa!}(\vartheta_{\beta,\tau,\gamma}(\kappa)\kappa!)=\sum_{\kappa=2}^{\infty}\frac{(\kappa)^2}{\kappa!}\ell(\kappa)<1
\end{aligned}
\tag{23}
$$

where

$$\ell(\kappa)=\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)(1)_\kappa}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\tag{24}$$

The series in (23) is transformed into a sum of twice the terms by employing the following relation:

$$\frac{\kappa^2}{(1)_\kappa}=\frac{\kappa}{(1)_{\kappa-1}}=\frac{1}{(1)_{\kappa-1}}+\frac{1}{(1)_{\kappa-2}}\tag{25}$$

Depending on $(1)_\kappa=\kappa!$ and $(1)_{\kappa-1}=(\kappa-1)!$, the estimate (23) becomes the next form:

$$
\begin{aligned}
\ell_1&\leq\sum_{\kappa=2}^{\infty}\frac{\kappa^2}{(1)_\kappa}=\sum_{\kappa=2}^{\infty}\left(\frac{1}{(1)_{\kappa-1}}+\frac{1}{(1)_{\kappa-2}}\right)\ell(\kappa)=\sum_{\kappa=2}^{\infty}\frac{\ell(\kappa)}{(1)_{\kappa-1}}+\frac{\ell(\kappa)}{(1)_{\kappa-2}}\\
&=\sum_{\kappa=2}^{\infty}\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_\kappa}{(1)_{\kappa-1}}\\
&\quad+\sum_{\kappa=2}^{\infty}\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_\kappa}{(1)_{\kappa-2}}\\
&=\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\left(\sum_{\kappa=1}^{\infty}\frac{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_{\kappa+1}}{(1)_\kappa}\right.\\
&\quad\left.+\sum_{\kappa=0}^{\infty}\frac{\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_{\kappa+2}}{(1)_\kappa}\right)
\end{aligned}
$$

by considering some properties of the gamma function in (13) and (12), we have

$$
\begin{aligned}
&=\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\left(\sum_{\kappa=1}^{\infty}\frac{\Gamma(\kappa+2)\Gamma\left(\frac{\kappa+\beta}{\gamma}+1\right)}{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1-\beta+\tau\right)}\frac{1}{(1)_\kappa}\right.\\
&\quad\left.+\sum_{\kappa=0}^{\infty}\frac{\Gamma(\kappa+3)\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1-\beta+\tau\right)}\frac{1}{(1)_\kappa}\right)
\end{aligned}
$$

and by using the Fox–Wright function, we can transform the estimate ℓ_1 at $z\to 1$,

$$= {}_2\Lambda_1\left(\begin{array}{c} (3,1),\left(1+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1}\right); \end{array} 1\right)$$
$$+ {}_2\Lambda_1\left(\begin{array}{c} (2,1),\left(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \\ \left(1-\beta+\upsilon+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \end{array} 1\right)$$
$$-\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)} < \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}.$$

We conclude from the above theorem that the operator $\Theta_z^{\beta,\tau,\gamma}f(z)$ maps preserve the property (univalent function) in class $f \in \mathcal{S}$ from a linear space to another. Further, the operator $\Theta_z^{\beta,\tau,\gamma}f(z)$ is univalent for $f \in \mathcal{S}$ for all $z \in \mathbb{U}$ in the open unit disk and $\Theta_z^{\beta,\tau,\gamma} : \mathcal{S} \to \mathcal{S}$.

Theorem 8 *Let the condition i as in the Theorem 7 is satisfied, then*

$${}_2\Lambda_1\left(\begin{array}{c} (2,1),\left(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \\ \left(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \end{array} 1\right) < 2\left(\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}\right).$$

then the operator maps a convex function f(z) into a univalent function that is $\Theta_z^{\beta,\tau,\gamma} : \mathcal{K} \to \mathcal{S}$.

Proof Presume that $f(z) \in \mathcal{K},\ z \in \mathcal{U}$ and the operator (17), such that

$$\Theta_z^{\beta,\tau,\gamma}f(z) = z + \sum_{\kappa=2}^{\infty} w_\kappa z^\kappa$$

where

$$w_\kappa := \vartheta_{\beta,\tau,\gamma}(\kappa)a_\kappa$$

and the function $\vartheta_{\beta,\tau,\gamma}$ is defined in inequality (18), satisfied the following condition in class $\mathcal{S}$ as follows:

$$\ell_2 ;= \sum_{\kappa=2}^{\infty} \kappa|w_\kappa| = \sum_{\kappa=2}^{\infty} \kappa\vartheta_{\beta,\tau,\gamma}(\kappa)|a_\kappa| < 1.$$

We know That the coefficient of a convex function belong to $\mathcal{S}$ is $|a_\kappa| < 1$. So we can get another estimate for ℓ_2 as follows,

$$\begin{aligned}\ell_2 &= \sum_{\kappa=2}^{\infty} \kappa\vartheta_{\beta,\tau,\gamma}|a_\kappa| \le \sum_{\kappa=2}^{\infty} \kappa^2\vartheta_{\beta,\tau,\gamma}(\kappa) \\ &= \sum_{\kappa=2}^{\infty} \frac{(\kappa)^2}{\kappa!}(\vartheta_{\beta,\tau,\gamma}(\kappa)\kappa!) = \sum_{\kappa=2}^{\infty} \frac{(\kappa)^2}{\kappa!}\ell(\kappa) < 1\end{aligned} \tag{26}$$

where

$$\ell(\kappa) = \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} \frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)(1)_\kappa}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}$$

where a_κ is Pochhammer symbol defined in (11),with the following relation

$$\frac{\kappa}{(1)_\kappa} = \frac{1}{(1)_{\kappa-1}}$$

and $(1)_\kappa = \kappa!$, then the estimate (26) become as the next form

$$\begin{aligned}\ell_2 &\le \sum_{\kappa=2}^{\infty} \frac{\kappa}{(1)_\kappa}\ell(\kappa) = \sum_{\kappa=2}^{\infty} \frac{1}{(1)_{\kappa-1}}\ell(\kappa) \\ &= \sum_{\kappa=2}^{\infty} \frac{\ell(\kappa)}{(1)_{\kappa-1}} = \sum_{\kappa=2}^{\infty} \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\beta+1}+1\right)} \\ &\quad \frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)} \frac{(1)_\kappa}{(1)_{\kappa-1}} \\ &= \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} \sum_{\kappa=1}^{\infty} \frac{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1-\beta+\tau\right)} \frac{(1)_{\kappa+1}}{(1)_\kappa}\end{aligned}$$

employ the properties of gamma function, we have

$$= \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} \sum_{\kappa=1}^{\infty} \frac{\Gamma(\kappa+2)\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1-\beta+\tau\right)} \frac{1}{(1)_\kappa}$$

then with the Fox–Wright function, we transform the estimate ℓ_1 at $z = 1$,

$$= \frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)} {}_2\Lambda_1\left(\begin{array}{c} (2,1),\left(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right); \\ \\ \left(1-\beta+\tau+\frac{\beta}{\eta+1},\frac{1}{\eta+1}\right); \end{array} 1\right) - 1 < 1$$

hence

$\Theta_z^{\beta,\tau,\gamma} : \mathcal{K} \to \mathcal{S}.$

We conclude from the above theorem that the maps preserve the property (univalent function) in class $\mathcal{S}$ from a linear space to another. □

Coefficients bound

Now, we study the coefficient bounds for the operator $\Theta_z^{\beta,\tau,\gamma} f(z)$, which is defined in (20), where the function $f(z)$ is in the class $\mathcal{A}(m)$, for all $z \in \mathbb{U}$. We also discuss the bounded coefficient in two subclasses $\mathcal{S}_\lambda^*(m)$ and $\mathcal{K}_\lambda(m)$, $(m = 1, 2, \ldots)$ of order λ in the open unit disk $\mathbb{U}$. In the first step, we are looking to prove that the operator $\Theta_z^{\beta,\tau,\gamma} f(z)$ in $\mathcal{S}_\lambda^*(m)$, and that by finding a coefficient bound.

Theorem 9 *Let $f(z) \in \mathcal{A}(m)$ given by (1) satisfy the condition (16). If*

$$\sum_{\kappa=m+1}^{\infty} (\kappa - \lambda)\,|a_\kappa| \le \frac{1-\lambda}{\vartheta_{\beta,\tau,\gamma}(m+1)} \quad (0 \le \lambda < 1) \tag{27}$$

where

$$\vartheta_{\beta,\tau,\gamma}(m+1) := \frac{\Gamma\left(\frac{\beta}{\gamma+1} + 1 - \beta + \tau\right)\Gamma\left(\frac{m+\beta}{\gamma+1} + 1\right)}{\Gamma\left(\frac{\beta}{\gamma+1} + 1\right)\Gamma\left(\frac{m+\beta}{\gamma+1} + 1 - \beta + \tau\right)},$$

Then the operator $\Theta_z^{\beta,\tau,\gamma} f(z) \in \mathcal{S}_\lambda^*(m)$ and satisfy the sharp result.

Proof By assuming that the function $f \in \mathcal{S}_\lambda^*(m)$, we obtain

$$\begin{aligned}\left|\frac{z(\Theta_z^{\beta,\tau,\gamma} f(z))'}{\Theta_z^{\beta,\tau,\gamma} f(z)} - 1\right| &= \left|\frac{z(\Theta_z^{\beta,\tau,\gamma} f(z))' - \Theta_z^{\beta,\tau,\gamma} f(z)}{\Theta_z^{\beta,\tau,\gamma} f(z)}\right| \\ &= \left|\frac{\sum_{\kappa=m+1}^{\infty} (\kappa-1)\,\vartheta_{\beta,\tau,\gamma}(\kappa) a_\kappa z^\kappa}{z + \sum_{\kappa=m+1}^{\infty} \vartheta_{\beta,\tau,\gamma}(\kappa) a_\kappa z^\kappa}\right| \\ &\le \frac{\sum_{\kappa=m+1}^{\infty} (\kappa-1)\,|\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa|\,|z|^{\kappa-1}}{1 - \sum_{\kappa=m+1}^{\infty} |\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa|\,|z|^{\kappa-1}}, \quad |z| < 1 \\ &\le \frac{\sum_{\kappa=m+1}^{\infty} (\kappa-1)\,|\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa|}{1 - \sum_{\kappa=m+1}^{\infty} |\vartheta_{\beta,\tau,\gamma}(\kappa)||a_\kappa|}\end{aligned} \tag{28}$$

we see in inequality (28) is bounded by $(1-\lambda)$, if it satisfy

$$\sum_{\kappa=m+1}^{\infty} (\kappa-1)\,|\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa| \le (1-\lambda)\left(1 - \sum_{\kappa=m+1}^{\infty} |\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa|\right),$$

By use the inequality $0 \le \vartheta_{\beta,\tau,\gamma}(\kappa) \le \vartheta_{\beta,\tau,\gamma}(m+1)$; for each $m+1 \le \kappa$ and for all $m = 1, 2, \ldots$, we have

$$\begin{aligned}&\sum_{\kappa=m+1}^{\infty} (\kappa-1)\,|\vartheta_{\beta,\tau,\gamma}(\kappa)|\,|a_\kappa| \\ &\quad \le \vartheta_{\beta,\tau,\gamma}(m+1) \sum_{\kappa=m+1}^{\infty} (\kappa-1)|a_\kappa| \\ &\quad \le (1-\lambda)\left(1 - \sum_{\kappa=m+1}^{\infty} |\vartheta_{\beta,\tau,\gamma}(\kappa)||a_\kappa|\right),\end{aligned}$$

which is on a par with

$$\sum_{\kappa=m+1}^{\infty} (\kappa-\lambda)|a_\kappa| \le \frac{1-\lambda}{\vartheta_{\beta,\tau,\gamma}(m+1)}, \tag{29}$$

hence

$$\mathcal{R}\left(\frac{z(\Theta_z^{\beta,\tau,\gamma} f(z))'}{\Theta_z^{\beta,\tau,\gamma} f(z)}\right) > \lambda.$$

From Theorem (9), we have a special case compared with the well-known results, which are reviewed in the next corollary.

Corollary 1 *Let $\Theta_z^{\beta,\tau,\gamma} f(z) \in \mathcal{S}_\lambda^*(m)$, for all $z \in \mathbb{U}$, with*

$$\sum_{\kappa=m+1}^{\infty} (\kappa-\lambda)\,|a_\kappa| \le \frac{1-\lambda}{\vartheta_{\beta,\tau,\gamma}(m+1)},$$

(i) *If $\lambda = 0, \beta = 1, \tau = 1, m = 1$ and $\gamma = 0$, we get*

$$\sum_{\kappa=2}^{\infty} \kappa|a_\kappa| \le 1$$

then $f(z) \in \mathcal{S}_0^*(1) \equiv \mathcal{S}^*$ (see [19]).

(ii) *If $\beta = 1, \tau = 1, m = 1$ and $\gamma = 0$, we get*

$$\sum_{\kappa=2}^{\infty} (\kappa-\lambda)|a_\kappa| \le (1-\lambda)$$

then $f(z) \in \mathcal{S}_\lambda^*(1) \equiv \mathcal{S}_\lambda^*$ (see [20]).

(iii) *If $\lambda = 0$, we get*

$$\sum_{\kappa=m+1}^{\infty} \kappa\,|a_\kappa| \le \frac{1}{\vartheta_{\beta,\tau,\gamma}(m+1)}. \tag{30}$$

then $\Theta_z^{\beta,\tau,\gamma} f(z) \in \mathcal{S}_0^*(m)$, (see [21]).

All these results are sharp.

Corollary 2 *Let the operator $\Theta_z^{\beta,\tau,\gamma} f(z) \in \mathcal{S}_\lambda^*(m)$. Then*

$$|a_{m+1}| \le \frac{(1-\lambda)\Gamma\left(\frac{\beta}{\gamma+1} + 1\right)\Gamma\left(\frac{m+\beta}{\gamma+1} + 1 - \beta + \tau\right)}{(m-\lambda+1)\Gamma\left(\frac{\beta}{\gamma+1} + 1 - \beta + \tau\right)\Gamma\left(\frac{m+\beta}{\gamma+1} + 1\right)}, \tag{31}$$

for $m = \{1, 2, 3, \ldots\}$.

Example 3 The function belongs to the class $\mathcal{S}^*_\lambda(m)$, is defined as

$$g_1(z) = z + \frac{(1-\lambda)\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{m+\beta}{\gamma+1}+1-\beta+\tau\right)}{(m-\lambda+1)\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{m+\beta}{\gamma+1}+1\right)} z^{m+1}.$$

We prove a bound coefficient in Theorem (10) by using similar methods in the starlike class.

Theorem 10 *Let the function* $f(z) \in \mathcal{A}(m)$ *and satisfied the condition* (16). *If*

$$\sum_{\kappa=m+1}^{\infty} \kappa(\kappa-\lambda)\,|a_\kappa| \leq \frac{1-\lambda}{\vartheta_{\beta,\tau,\gamma}(m+1)} \quad m = 1, 2, \ldots. \tag{32}$$

Then $f \in \mathcal{K}_\lambda(m)$, $\lambda\,(0 \leq \lambda < 1)$, *this result is sharp.*

Corollary 3 *Let the operator* $\Theta_z^{\beta,\tau,\gamma} f(z) \in \mathcal{K}_\lambda(m)$. *Then*

$$|a_{m+1}| \leq \frac{(1-\lambda)\Gamma(\frac{\beta}{\gamma+1}+1)\Gamma(\frac{m+\beta}{\gamma+1}+1-\beta+\tau)}{(m+1)(m-\lambda+1)\Gamma(\frac{\beta}{\gamma+1}+1-\beta+\tau)\Gamma(\frac{m+\beta}{\gamma+1}+1)},$$

for $\kappa \in \{2, 3, \ldots\}$.

Example 4 The equality (32) is realized by the function

$$g_2(z) = z + \frac{(1-\lambda)\Gamma\left(\frac{\beta}{\gamma+1}+1\right)\Gamma\left(\frac{m+\beta}{\gamma+1}+1-\beta+\tau\right)}{(m+1)(m-\lambda+1)\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)\Gamma\left(\frac{m+\beta}{\gamma+1}+1\right)} z^{m+1}.$$

Conclusion

All results of the present work are valid in open unit disk U with respect to the fractional calculus in a complex domain. We defined a normalized fractional differential operator in the concept of the generalized Tremblay operator. Moreover, we assumed sufficient conditions for this operator to become starlike and convex functions. Finally, univalency and convolution properties are discussed.

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